KLINE

15
Coordinate Geometry

. . . I have resolved to quit only abstract geometry, that is to say, the consideration of questions which *serve only to exercise the mind*, and this, in order to study another kind of geometry, which has for its object the explanation of the phenomena of nature. RENÉ DESCARTES

1. *The Motivation for Coordinate Geometry*

Fermat and Descartes, the two men primarily responsible for the next major creation in mathematics, were, like Desargues and his followers, concerned with general methods for studying curves. But Fermat and Descartes were very much involved in scientific work, keenly aware of the need for quantitative methods, and impressed with the power of algebra to supply that method. And so Fermat and Descartes turned to the application of algebra to the study of geometry. The subject they created is called coordinate, or analytic, geometry; its central idea is the association of algebraic equations with curves and surfaces. This creation ranks as one of the richest and most fruitful veins of thought ever struck in mathematics.

That the needs of science and an interest in methodology motivated both Fermat and Descartes is beyond doubt. Fermat's contributions to the calculus such as the construction of tangents to curves and the calculation of maxima and minima, were, as we shall see more clearly in connection with the history of the calculus, designed to answer scientific problems; he was also a first-rate contributor to optics. His interest in methodology is attested to by an explicit statement in his brief book, *Ad Locos Planos et Solidos Isagoge* (Introduction to Plane and Solid Loci[1]), written in 1629 but published by 1637.[2] He says there that he sought a universal approach to problems involving curves. As for Descartes, he was one of the greatest seventeenth-century scientists, and he made methodology a prime objective in all of his work.

1. Fermat uses these terms in the sense explained by Pappus. See Chap. 8, sec. 2.
2. *Œuvres*, 1, 91–103.

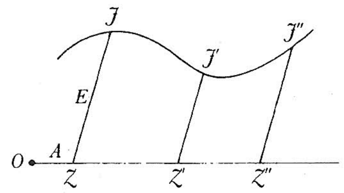

Figure 15.1

2. *The Coordinate Geometry of Fermat*

In his work on the theory of numbers, Fermat started with Diophantus. His work on curves began with his study of the Greek geometers, notably Apollonius, whose lost book, *On Plane Loci*, he, among others, had reconstructed. Having contributed to algebra, he was prepared to apply it to the study of curves, which he did in *Ad Locos*. He says that he proposed to open up a general study of loci, which the Greeks had failed to do. Just how Fermat's ideas on coordinate geometry evolved is not known. He was familiar with Vieta's use of algebra to solve geometric problems, but it is more likely that he translated Apollonius' results directly into algebraic form.

He considers any curve and a typical point J on it (Fig. 15.1). The position of J is fixed by a length A, measured from a point O on a base line to a point Z, and the length E from Z to J. Thus Fermat uses what we call oblique coordinates, though no y-axis appears explicitly and no negative coordinates are used. His A and E are our x and y.

Fermat had stated earlier his general principle: "Whenever in a final equation two unknown quantities are found we have a locus, the extremity of one of these describing a line straight or curved." Thus the extremities $J, J', J'', \ldots$ of E in its various positions describe the "line." His unknown quantities, A and E, are really variables or, one can say, the equation in A and E is indeterminate. Here Fermat makes use of Vieta's idea of having a letter stand for a class of numbers. Fermat then gives various algebraic equations in A and E and states what curves they describe. Thus he writes "*D in A aequetur B in E*" (in our notation, $Dx = By$) and states that this represents a straight line. He also gives (in our notation) the more general equation $d(a - x) = by$ and affirms that this too represents a straight line. The equation "*B quad. – A quad. aequetur E quad.*" (in our notation, $B^2 - x^2 = y^2$) represents a circle. Similarly (in our notation), $a^2 - x^2 = ky^2$ represents an ellipse; $a^2 + x^2 = ky^2$ and $xy = a$ represent hyperbolas; and $x^2 = ay$ represents a parabola. Since Fermat did not use negative coordinates, his equations could not represent the full curve that he said they described. He did appreciate that one can translate and rotate axes, because he gives more complicated second-degree equations and states the simpler forms to which they can be reduced. In fact, he affirms that an equation of the first degree in A and E has a straight-line locus and all second degree equations in A and E have conics as their loci. In his *Methodus ad Disquirendam Maximam et Minimam* (Method of Finding Maxima and Minima, 1637),[3] he introduced the curves of $y = x^n$ and $y = x^{-n}$.

3. *René Descartes*

Descartes was the first great modern philosopher, a founder of modern biology, a first-rate physicist, and only incidentally a mathematician. However, when a man of his power of intellect devotes even part of his time to a subject, his work cannot but be significant.

He was born in La Haye in Touraine on March 31, 1596. His father, a moderately wealthy lawyer, sent him at the age of eight to the Jesuit school of La Flèche in Anjou. Because he was of delicate health, he was allowed to spend the mornings in bed, during which time he worked. He followed this custom throughout his life. At sixteen he left La Flèche and at twenty he was graduated from the University of Poitiers as a lawyer and went to Paris. There he met Mydorge and Father Marin Mersenne and spent a year with them in the study of mathematics. However, Descartes became restless and entered the army of Prince Maurice of Orange in 1617. During the next nine years he alternated between service in several armies and carousing in Paris, but throughout this period continued to study mathematics. His ability to solve a problem that had been posted on a billboard in Breda in the Netherlands as a challenge convinced him that he had mathematical ability and he began to think seriously in this subject. He returned to Paris and, having become excited by the power of the telescope, secluded himself to study the theory and construction of optical instruments. In 1628 he moved to Holland to secure a quieter and freer intellectual atmosphere. There he lived for twenty years and wrote his famous works. In 1649 he was invited to instruct Queen Christina of Sweden. Tempted by the honor and the glamor of royalty, he accepted. He died there of pneumonia in 1650.

His first work, *Regulae ad Directionem Ingenii* (Rules for the Direction of the Mind),[4] was written in 1628 but published posthumously. His next major work was *Le Monde* (System of the World, 1634), which contains a cosmological theory of vortices to explain how the planets are kept in motion and in their paths around the sun. However, he did not publish it for fear of persecution by the Church. In 1637 he published his *Discours de la méthode pour bien conduire sa raison, et chercher la vérité dans les sciences.*[5] This book, a classic of literature and philosophy, contains three famous appendices, *La Géométrie*, *La Dioptrique*, and *Les Météores*. *La Géométrie*, which is the only book Descartes wrote on mathematics, contains his ideas on coordinate geometry and algebra, though he did communicate many other ideas on mathematics in numerous letters. The *Discours* brought him great fame immediately. As time passed, both he and his public became more impressed with his work. In 1644 he published *Principia Philosophiae*, which is devoted to physical science and especially to the laws of motion and the theory of vortices. It contains material from his *System*, which he believed he had now made more acceptable to the Church. In 1650 he published *Musicae Compendium*.

Descartes's scientific ideas came to dominate the seventeenth century. His teachings and writings became popular even among non-scientists because he presented them so clearly and attractively. Only the Church rejected him. Actually Descartes was devout, and happy to have (as he believed) established the existence of God. But he had taught that the Bible was not the source of scientific knowledge, that reason alone sufficed to establish the existence of God, and that man should accept only what he could understand. The Church reacted to these teachings by putting his books on the *Index of Prohibited Books* shortly after his death and by preventing a funeral oration on the occasion of his interment in Paris.

Descartes approached mathematics through three avenues, as a philosopher, as a student of nature, and as a man concerned with the uses of science. It is difficult and perhaps artificial to try to separate these three lines of thought. He lived when the Protestant-Catholic controversy was at its height and when science was beginning to reveal laws of nature that challenged major religious doctrines. Hence Descartes began to doubt all the knowledge he had acquired at school. As early as the conclusion of his course of study at La Flèche, he decided that his education had advanced only his perplexity. He found himself so beset with doubts that he was convinced he had progressed no further than to recognize his ignorance. And yet, because he had been in one of the most celebrated schools in Europe, and because he believed he had not been an inferior student, he felt justified in doubting whether there was any sure body of knowledge anywhere. He then pondered the question: How do we know anything?

He soon decided that logic in itself was barren: "As for Logic, its syllogisms and the majority of its other precepts are of avail rather in the communication of what we already know, or . . . even in speaking without judgment of things of which we are ignorant, than in the investigation of the unknown." Logic, then, did not supply the fundamental truths.

But where were these to be found? He rejected the current philosophy, largely Scholastic, which, though appealing, seemed to have no clear-cut foundations and employed reasoning that was not always irreproachable.

3. *Œuvres*, 1, 133–79; 3, 121–56.
4. Published in Dutch in 1692; *Œuvres*, 10, 359–469.
5. *Œuvres*, 6, 1–78.

Philosophy, he decided, afforded merely "the means of discoursing with an appearance of truth on all matters." Theology pointed out the path to heaven and he aspired to go there as much as any man, but was the path correct?

The method of establishing truths in all fields came to him, he says, in a dream, on November 10, 1619, when he was on one of his military campaigns; it was the method of mathematics. Mathematics appealed to him because the proofs based on its axioms were unimpeachable and because authority counted for naught. Mathematics provided the method of achieving certainties and effectively demonstrating them. Moreover, he saw clearly that the method of mathematics transcended its subject matter. He says, "It is a more powerful instrument of knowledge than any other that has been bequeathed to us by human agency, as being the source of all others." In the same vein he continues:

> . . . All the sciences which have for their end investigations concerning order and measure are related to mathematics, it being of small importance whether this measure be sought in numbers, forms, stars, sounds, or any other object; that accordingly, there ought to exist a general science which should explain all that can be known about order and measure, considered independently of any application to a particular subject, and that, indeed, this science has its own proper name, consecrated by long usage, to wit, mathematics. And a proof that it far surpasses in facility and importance the sciences which depend upon it is that it embraces at once all the objects to which these are devoted and a great many others besides. . . .

And so he concluded that "The long chains of simple and easy reasonings by means of which geometers are accustomed to reach the conclusions of their most difficult demonstrations had led me to imagine that all things to the knowledge of which man is competent are mutually connected in the same way."

From his study of mathematical method he isolated in his *Rules for the Direction of the Mind* the following principles for securing exact knowledge in any field. He would accept nothing as true that was not so clear and distinct in his own mind as to exclude all doubt; he would divide difficulties into smaller ones; he would proceed from the simple to the complex; and, lastly, he would enumerate and review the steps of his reasoning so thoroughly that nothing could be omitted.

With these essentials of method, which he distilled from the practice of mathematicians, Descartes hoped to solve problems in philosophy, physics, anatomy, astronomy, mathematics, and other fields. Although he did not succeed in this ambitious program, he did make remarkable contributions to philosophy, science, and mathematics. The mind's immediate apprehension of basic, clear, and distinct truths, this intuitive power, and the deduction of consequences are the essence of his philosophy of knowledge. Purported knowledge otherwise obtained should be rejected as suspect of error and dangerous. The three appendices to his *Discours* were intended to show that his method is effective; he believed that he had shown this.

Descartes inaugurated modern philosophy. We cannot pursue his system except to note a few points relevant to mathematics. In philosophy he sought as axioms truths so clear to him that he could accept them readily. He finally decided on four: (a) *cogito, ergo sum* (I think, therefore I am); (b) each phenomenon must have a cause; (c) an effect cannot be greater than its cause; (d) the mind has innate in it the ideas of perfection, space, time, and motion. The idea of perfection, of a perfect being, could not be derived from or created by the imperfect mind of man in view of axiom (c). It could be obtained only from a perfect being. Hence God exists. Since God would not deceive us, we can be sure that the axioms of mathematics, which are clear to our intuitions, and the deductions we make from them by purely mental processes, really apply to the physical world and so are truths. It follows, then, that God must have established nature according to mathematical laws.

As for mathematics itself, he believed that he had distinct and clear mathematical ideas, such as that of a triangle. These ideas did exist and were eternal and immutable. They did not depend on his thinking them or not. Thus mathematics had an external, objective existence.

Descartes's second major interest, shared by most thinkers of his age, was the understanding of nature. He devoted many years to scientific problems and even experimented extensively in mechanics, hydrostatics, optics, and biology. His theory of vortices was the dominant cosmological theory of the seventeenth century. He is the founder of the philosophy of mechanism—that all natural phenomena, including the functioning of the human body, reduce to motions obeying the laws of mechanics—though Descartes exempted the soul. Optics, and the design of lenses in particular, was of special interest to him; part of *La Géométrie* is devoted to optics, as is *La Dioptrique*. Descartes shares with Willebrord Snell the honor of discovering the correct law of refraction of light. As in philosophy, his work in science was basic and revolutionary.

Also important in Descartes's scientific work is his emphasis on putting the fruits of science to use (Chap. 11, sec. 5). In this attitude he breaks clearly and openly with the Greeks. To master nature for the good of man, he pursued many scientific problems. And, being impressed with the power of mathematics, he naturally sought to use that subject; for him it was not contemplative discipline but a constructive and useful science. Unlike Fermat, he cared little for its beauty and harmony; he did not value pure mathematics. He says that mathematical method applied only to mathematics is without value because it is not a study of nature. Those who cultivate mathematics for its own sake are idle searchers given to a vain play of the spirit.

4. *Descartes's Work in Coordinate Geometry*

Having decided that method was important and that mathematics could be effectively employed in scientific work, Descartes turned to the application of method to geometry. Here his general interest in method and his particular knowledge of algebra joined forces. He was disturbed by the fact that every proof in Euclidean geometry called for some new, often ingenious, approach. He explicitly criticized the geometry of the ancients as being too abstract, and so much tied to figures "that it can exercise the understanding only on condition of greatly fatiguing the imagination." The algebra that he found prevalent he also criticized because it was so completely subject to rules and formulas "that there results an art full of confusion and obscurity calculated to hamper instead of a science fitted to improve the mind." Descartes proposed, therefore, to take all that was best in geometry and algebra and correct the defects of one with the help of the other.

Actually it was the use of algebra in geometry that he undertook to exploit. He saw fully the power of algebra and its superiority over the Greek geometrical methods in providing a broad methodology. He also stressed the generality of algebra and its value in mechanizing the reasoning processes and minimizing the work in solving problems. He saw its potential as a universal science of method. The product of his application of algebra to geometry was *La Géométrie*.

Though in this book Descartes used the improvements in algebraic notation already noted in Chapter 13, the essay is not easy reading. Much of the obscurity was deliberate; Descartes boasted that few mathematicians in Europe would understand his work. He indicated the constructions and demonstrations, leaving it to others to fill in the details. In one of his letters he compares his writing to that of an architect who lays the plans and prescribes what should be done but leaves the manual work to the carpenters and bricklayers. He says also, "I have omitted nothing inadvertently but I have foreseen that certain persons who boast that they know everything would not miss the opportunity of saying that I have written nothing that they did not already know, were I to make myself sufficiently intelligible for them to understand me." He gave other reasons in *La Géométrie*, such as not wishing to deprive his readers of the pleasure of working things out for themselves. Many explanatory commentaries were written to make Descartes's book clear.

His ideas must be inferred from a number of examples worked out in the book. He says that he omits the demonstration of most of his general statements because if one takes the trouble to examine systematically these

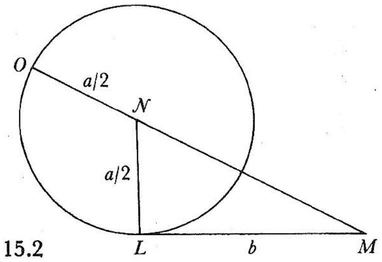

Figure 15.2

examples, the demonstrations of the general results will become apparent, and it is of more value to learn them in that way.

He begins in *La Géométrie* with the use of algebra to solve geometrical construction problems in the manner of Vieta; only gradually does the idea of the equation of a curve emerge. He points out first that geometrical constructions call for adding, subtracting, multiplying, and dividing lines and taking the square root of particular lines. Since all of these operations also exist in algebra, they can be expressed in algebraic terms.

In tackling a given problem, Descartes says we must suppose the solution of the problem already known and represent with letters all the lines, known and unknown, that seem necessary for the required construction. Then, making no distinction between known and unknown lines, we must "unravel" the difficulty by showing in what way the lines are related to each other, aiming at expressing one and the same quantity in two ways. This gives an equation. We must find as many equations as there are unknown lines. If several equations remain, we must combine them until there remains a single unknown line expressed in terms of known lines. Descartes then shows how to construct the unknown line by utilizing the fact that it satisfies the algebraic equation.

Thus, suppose a geometric problem leads to finding an unknown length x, and after algebraic formulation x is found to satisfy the equation $x^2 = ax + b^2$ where a and b are known lengths. Then we know by algebra that

$$x = \frac{a}{2} + \sqrt{\frac{a^2}{4} + b^2}. \tag{1}$$

(Descartes ignored the second root, which is negative.) Descartes now gives a construction for x. He constructs the right triangle NLM (Fig. 15.2) with $LM = b$ and $NL = a/2$, and prolongs MN to O so that $NO = NL = a/2$. Then the solution x is the length OM. The proof that OM is the correct length is not given by Descartes but it is immediately apparent for

$$OM = ON + MN = \frac{a}{2} + \sqrt{\frac{a^2}{4} + b^2}.$$

Thus the expression (1) for x, which was obtained by solving an algebraic equation, indicates the proper construction for x.

In the first half of Book I, Descartes solves only classical geometric construction problems with the aid of algebra. This is an application of algebra to geometry, but not analytic geometry in our present sense. The problems thus far are what one might call determinate construction problems because they lead to a unique length. He considers next indeterminate construction problems, that is, problems in which there are many possible lengths that serve as answers. The endpoints of the many lengths fill out a curve; and here Descartes says, "It is also required to discover and trace the curve containing all such points." This curve is described by the final indeterminate equation expressing the unknown lengths y in terms of the arbitrary lengths x. Moreover, Descartes stresses that for each x, y satisfies a determinate equation and so can be constructed. If the equation is of the first or second degree, y can be constructed by the methods of Book I, using only lines and circles. For higher-degree equations, he says he will show in Book III how y can be constructed.

Descartes uses the problem of Pappus (Chap. 5, sec. 7) to illustrate what happens when a problem leads to one equation in two unknowns. This problem, which had not been solved in full generality, is as follows: Given the position of three lines in a plane, find the position of all points (the locus) from which we can construct lines, one to each of the given lines and making a known angle with each of these given lines (the angle may be different from line to line), such that the rectangle contained by two of the constructed lines has a given ratio to the square on the third constructed line; if there are four given lines, then the constructed lines, making given angles with the given lines, must be such that the rectangle contained by two must have a given ratio to the rectangle contained by the other two; if there are five given lines, then the five constructed lines, each making a given angle with one of the given lines, must be such that the product of three of them has a given ratio to the product of the remaining two. The condition on the locus when there are more than five given lines is an obvious extension of the above.

Pappus had declared that when three or four lines are given, the locus is a conic section. In Book II Descartes treats the Pappus problem for the case of four lines. The given lines (Fig. 15.3) are AG, GH, EF, and AD. Consider a point C and the four lines from C to each of the four given lines and making a specified angle with each of the four given lines. The angle can be different from one line to another. Let us denote the four lines by CP, CQ, CR, and CS. It is required to find the locus of C satisfying the condition $CP \cdot CR = CS \cdot CQ$.

Descartes denotes AP by x and PC by y. By simple geometric considerations, he obtains the values of CR, CQ, and CS in terms of known

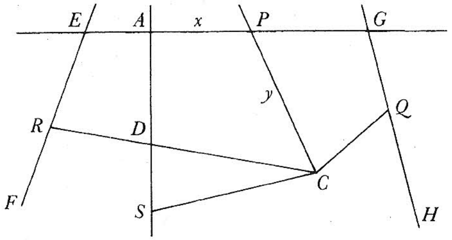

Figure 15.3

quantities. He uses these values to form $CP \cdot CR = CS \cdot CQ$ and obtains a second degree equation in x and y of the form

$$y^2 = Ay + Bxy + Cx + Dx^2 \qquad (2)$$

where A, B, C, and D are simple algebraic expressions in terms of the known quantities. Now Descartes points out that if we select any value of x we have a quadratic equation for y that can be solved for y; and then y can be constructed by straightedge and compass as he has shown in Book I. Hence if one takes an infinite number of values for x, one obtains an infinite number of values for y and hence an infinite number of points C. The locus of all these points C is a curve whose equation is (2).

What Descartes has done is to set up one line (AG in the above figure) as a base line with an origin at the point A. The x-values are then lengths along this line, and the y-values are lengths that start at this base line and make a fixed angle with it. This coordinate system is what we now call an oblique system. Descartes's x and y stand for positive numbers only; yet his equations cover portions of curves in other than what we would call the first quadrant. He simply assumes that the locus lies primarily in the first quadrant and makes passing reference to what might happen elsewhere. That there is a length for each positive real number is assumed unconsciously.

Having arrived at the idea of the equation of a curve, Descartes now develops it. It is easily demonstrated, he asserts, that the degree of a curve is independent of the choice of the reference axis; he advises choosing this axis so that the resulting equation is as simple as possible. In another great stride, he considers two different curves, expresses their equations with respect to the same reference axis, and finds the points of intersection by solving the equations simultaneously.

Also in Book II, Descartes considers critically the Greek distinctions among plane, solid, and linear curves. The Greeks had said plane curves were those constructible by straightedge and compass; the solid curves were the conic sections; and the linear curves were all the others, such as the conchoid, spiral, quadratrix, and cissoid. The linear curves were also called mechanical by the Greeks because some special mechanism was required to construct them. But, Descartes says, even the straight line and circle require some instrument. Nor can the accuracy of the mechanical construction matter, because in mathematics only the reasoning counts. Possibly, he continues, the ancients objected to linear curves because they were insecurely defined. On these grounds, Descartes rejects the idea that only the curves constructible with straightedge and compass[6] are legitimate and even proposes some new curves generated by mechanical constructions. He concludes with the highly significant statement that geometric curves are those that can be expressed by a unique algebraic equation (of finite degree) in x and y. Thus Descartes accepts the conchoid and cissoid. All other curves, such as the spiral and the quadratrix, he calls mechanical.

Descartes's insistence that an acceptable curve is one that has an algebraic equation is the beginning of the elimination of constructibility as a criterion of existence. Leibniz went farther than Descartes. Using the words "algebraic" and "transcendental" for Descartes's terms "geometrical" and "mechanical," he protested the requirement that a curve must have an algebraic equation.[7] Actually Descartes and his contemporaries ignored the requirement and worked just as enthusiastically with the cycloid, the logarithmic curve, the logarithmic spiral ($\log \rho = a\theta$), and other non-algebraic curves.

In broadening the concept of admissible curves, Descartes made a major step. He not only admitted curves formerly rejected but opened up the whole field of curves, because, given any algebraic equation in x and y, one can find its curve and so obtain totally new curves. In *Arithmetica Universalis* Newton says (1707), "But the Moderns advancing yet much further [than the plane, solid and linear loci of the Greeks] have received into Geometry all Lines that can be expressed by Equations."

Descartes next considers the classes of geometric curves. Curves of the first and second degree in x and y are in the first and simplest class. Descartes says, in this connection, that the equations of the conic sections are of the second degree, but does not prove this. Curves whose equations are of the third and fourth degree constitute the second class. Curves whose equations are of the fifth and sixth degree are of the third class and so on. His reason for grouping third and fourth, as well as fifth and sixth degree curves, is that he believed the higher one in each class could be reduced to the lower, as the solution of quartic equations could be effected by the solution of cubics. This belief was of course incorrect.

The third book of *La Géométrie* returns to the theme of Book I. Its objective is the solution of geometric construction problems, which, when formulated algebraically, lead to determinate equations of third and higher degree and which, in accordance with the algebra, call for the conic sections

6. Compare the discussion in Chap. 8, sec. 2.

7. *Acta Erud.*, 1684, pp. 470, 587; 1686, p. 292 = *Math. Schriften*, 5, 127, 223, 226.

and higher-degree curves. Thus Descartes considers the construction problem of finding the two mean proportionals between two given quantities a and q. The special case when $q = 2a$ was attempted many times by the classical Greeks and was important because it is a way to solve the problem of doubling the cube. Descartes proceeds as follows: Let z be one of these mean proportionals; then z^2/a must be the second, for we must have

$$\frac{a}{z} = \frac{z}{z^2/a} = \frac{z^2/a}{z^3/a^2}.$$

Then, if we take z^3/a^2 to be q, we have the equation z must satisfy. Hence, given q and a, we must find z such that

$$z^3 = a^2q, \qquad (3)$$

or, we must solve a cubic equation. Descartes now shows that such quantities z and z^2/a can be obtained by a geometrical construction that utilizes a parabola and a circle.

As the construction is described by Descartes, seemingly no coordinate geometry is involved. However, the parabola is not constructible with straightedge and compass, except point by point, and so one must use the equation to plot the curve accurately.

Descartes does *not* obtain z by writing the equations in x and y of circle and parabola and finding the coordinates of the point of intersection by solving equations simultaneously. In other words, he is not solving equations graphically in our sense. Rather he uses purely geometric constructions (except for supposing that a parabola can be drawn), the knowledge of the fact that z satisfies an equation, and the geometric properties of the circle and parabola (which can be more readily seen through their equations). Descartes does here just what he did in Book I, except that he is now solving geometric construction problems in which the unknown length satisfies a third or higher-degree equation instead of a first or second degree equation. His solution of the purely algebraic aspect of the problem and the subsequent construction is practically the same one the Arabs gave, except that he was able to use the equations of the conic sections to deduce facts about the curves and to draw them.

Descartes not only wished to show how some solid problems could be solved with the aid of algebra and the conic sections but was interested in classifying problems so that one would know what they involved and how to go about solving them. His classification is based on the degree of the algebraic equation to which one is led when the construction problem is formulated algebraically. If that degree is one or two, then the construction can be performed with straight line and circle. If the degree is three or four,

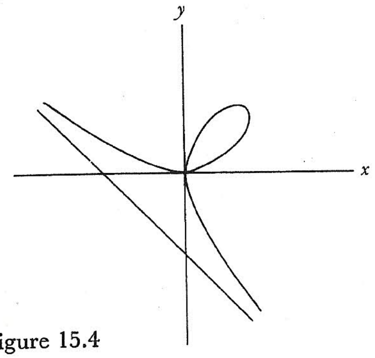

Figure 15.4

the conic sections must be employed. He does affirm, incidentally, that all cubic problems can be reduced to trisecting the angle and doubling the cube and that no cubic problems can be solved without the use of a curve more complex than the circle. If the degree of the equation is higher than four, curves more complicated than the conic sections may be required to perform the construction.

Descartes also emphasized the degree of the equation of a curve as the measure of its simplicity. One should use the simplest curve, that is, the lowest degree possible, to solve a construction problem. The emphasis on the degree of a curve became so strong that a complicated curve such as the folium of Descartes (Fig. 15.4), whose equation is $x^3 + y^3 - 3axy = 0$, was considered simpler than $y = x^4$.

What is far more significant than Descartes's insight into construction problems and their classification is the importance he assigned to algebra. This key makes it possible to recognize the typical problems of geometry and to bring together problems that in geometrical form would not appear to be related at all. Algebra brings to geometry the most natural principles of classification and the most natural hierarchy of method. Not only can questions of solvability and geometrical constructibility be decided elegantly, quickly, and fully from the parallel algebra, but without it they cannot be decided at all. Thus, system and structure were transferred from geometry to algebra.

Part of Book II of *La Géométrie* as well as *La Dioptrique* Descartes devoted to optics, using coordinate geometry as an aid. He was very much concerned with the design of lenses for the telescope, microscope, and other optical instruments because he appreciated the importance of these instruments for astronomy and biology. His *Dioptrique* takes up the phenomenon of refraction. Kepler and Alhazen before him had noted that the belief that the angle of refraction is proportional to the angle of incidence, the proportionality constant being dependent on the medium doing the refracting, was

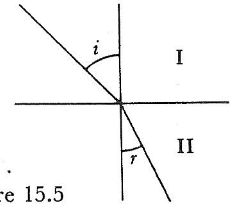

Figure 15.5

incorrect for large angles, but they did not discover the true law. Before 1626 Willebrord Snell discovered but did not publish the correct relationship,

$$\frac{\sin i}{\sin r} = \frac{v_1}{v_2},$$

where v_1 is the velocity of light in the first medium (Fig. 15.5) and v_2 the velocity in the medium into which the light passes. Descartes gave this same law in 1637 in the *Dioptrique*. There is some question as to whether he discovered it independently. His argument was wrong, and Fermat immediately attacked both the law and the proof. A controversy arose between them which lasted ten years. Fermat was not satisfied that the law was correct, until he derived it from his Principle of Least Time (Chap. 24, sec. 3).

In *La Dioptrique*, after describing the operation of the eye, Descartes considers the problem of designing properly focusing lenses for telescopes, microscopes, and spectacles. It was well known even in antiquity that a spherical lens will not cause parallel rays or rays diverging from a source S to focus on one point. Hence the question was open as to what shape would so focus the incoming rays. Kepler had suggested that some conic section would serve. Descartes sought to design a lens that would focus the rays perfectly.

He proceeded to solve the general problem of what surface should separate two media such that light rays starting from one point in the first medium would strike the surface, refract into the second medium, and there converge to one point. He discovered that the curve generating the desired surface of revolution is an oval, now known as the oval of Descartes. This curve and its refracting properties are discussed in *La Dioptrique*, and the discussion is supplemented in Book II of *La Géométrie*.

The modern definition is that the curve is the locus of points M satisfying the condition

$$FM \pm nF'M = 2a$$

where F and F' are fixed points, $2a$ is any real number larger than FF', and n is any real number. If $n = 1$ the curve becomes an ellipse. In the general case, the equation of the oval is of the fourth degree in x and y, and the curve

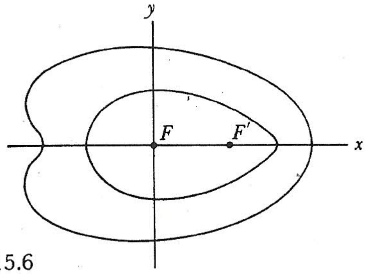

Figure 15.6

consists of two closed, distinct portions (Fig. 15.6) without common point and one inside the other. The inner curve is convex like an ellipse and the outer one can be convex or may have points of inflection, as in the figure.

As we can now see, Descartes's approach to coordinate geometry differs profoundly from Fermat's. Descartes criticized and proposed to break with the Greek tradition, whereas Fermat believed in continuity with Greek thought and regarded his work in coordinate geometry only as a reformulation of the work of Apollonius. The real discovery—the power of algebraic methods—is Descartes's; and he realized he was supplanting the ancient methods. Though the idea of equations for curves is clearer with Fermat than with Descartes, Fermat's work is primarily a technical achievement that completes the work of Apollonius and uses Vieta's idea of letters to represent classes of numbers. Descartes's methodology is universally applicable and potentially applies to the transcendental curves, too.

Despite these significant differences in approach to coordinate geometry and in goals, Descartes and Fermat became embroiled in controversy as to priority of discovery. Fermat's work was not published until 1679; however, his discovery of the basic ideas of coordinate geometry in 1629 predates Descartes's publication of *La Géométrie* in 1637. Descartes was by this time fully aware of many of Fermat's discoveries, but he denied having learned his ideas from Fermat. Descartes's ideas on coordinate geometry, according to the Dutch mathematician Isaac Beeckman (1588–1637), went back to 1619; and furthermore, there is no question about the originality of many of his basic ideas in coordinate geometry.

When *La Géométrie* was published, Fermat criticized it because it omitted ideas such as maxima and minima, tangents to curves, and the construction of solid loci, which, he had decided, merited the attention of all geometers. Descartes in turn said Fermat had done little, in fact no more than could be easily arrived at without industry or previous knowledge, whereas he himself had used a full knowledge of the nature of equations, which he had expounded in the third book of *La Géométrie*. Descartes referred sarcastically to Fermat as *vostre Conseiller De Maximis et Minimis* and said Fermat was indebted

to him. Roberval, Pascal, and others sided with Fermat, and Mydorge and Desargues sided with Descartes. Fermat's friends wrote bitter letters against Descartes. Later the attitudes of the two men toward each other softened, and in a work of 1660, Fermat, while calling attention to an error in *La Géométrie*, declared that he admired that genius so much that even when he made mistakes Descartes's work was worth more than that of others who did correct things. Descartes had not been so generous.

The emphasis placed by posterity on *La Géométrie* was not what Descartes had intended. While the salient idea for the future of mathematics was the association of equation and curve, for Descartes this idea was just a means to an end—the solution of geometric construction problems. Fermat's emphasis on the equations of loci is, from the modern standpoint, more to the point. The geometric construction problems that Descartes stressed in Books I and III have dwindled in importance, largely because construction is no longer used, as it was by the Greeks, to establish existence.

One portion of Book III has also found a permanent place in mathematics. Since Descartes solved geometric construction problems by first formulating them algebraically, solving the algebraic equations, and then constructing what the solutions called for, he gathered together work of his own and of others on the theory of equations that might expedite their solution. Because algebraic equations continued to arise in hundreds of different contexts having nothing to do with geometrical construction problems, this theory of equations has become a basic part of elementary algebra.

5. *Seventeenth-Century Extensions of Coordinate Geometry*

The main idea of coordinate geometry—the use of algebraic equations to represent and study curves—was not eagerly seized upon by mathematicians for many reasons. Fermat's book, the *Ad Locos*, though circulated among friends, was not published until 1679. Descartes's emphasis on the solution of geometric construction problems obscured the main idea of equation and curve. In fact, many of his contemporaries thought of coordinate geometry primarily as a tool for solving the construction problems. Even Leibniz spoke of Descartes's work as a regression to the ancients. Descartes himself did realize that he had contributed far more than a new method of solving construction problems. In the introduction to *La Géométrie* he says, "Moreover, what I have given in the second book on the nature and properties of curved lines, and the method of examining them, is, it seems to me, as far beyond the treatment of ordinary geometry as the rhetoric of Cicero is beyond the a, b, c of children." Nevertheless, the uses he made of the equations of the curves, such as solving the Pappus problem, finding normals to curves, and obtaining properties of the ovals, were far overshadowed by the attention given to the construction problems. Another reason for the slow spread of analytic geometry was Descartes's insistence on making his presentation difficult to follow.

In addition many mathematicians objected to confounding algebra and geometry, or arithmetic and geometry. This objection had been voiced even in the sixteenth century, when algebra was on the rise. For example, Tartaglia insisted on the distinction between the Greek operations with geometrical objects and operations with numbers. He reproached a translator of Euclid for using interchangeably *multiplicare* and *ducere*. The first belongs to numbers, he says, and the second to magnitude. Vieta, too, considered the sciences of number and of geometric magnitudes as parallel but distinct. Even Newton, in his *Arithmetica Universalis*, objected to confounding algebra and geometry, though he contributed to coordinate geometry and used it in the calculus. He says,[8]

> Equations are expressions of arithmetical computation and properly have no place in geometry except insofar as truly geometrical quantities (that is, lines, surfaces, solids and proportions) are thereby shown equal, some to others. Multiplications, divisions and computations of that kind have been recently introduced into geometry, unadvisedly and against the first principles of this science. . . . Therefore these two sciences ought not to be confounded, and recent generations by confounding them have lost that simplicity in which all geometrical elegance consists.

A reasonable interpretation of Newton's position is that he wanted to keep algebra out of elementary geometry but did find it useful to treat the conics and higher-degree curves.

Still another reason for the slowness with which coordinate geometry was accepted was the objection to the lack of rigor in algebra. We have already mentioned Barrow's unwillingness to accept irrational numbers as more than symbols for continuous geometrical magnitudes (Chap. 13, sec. 2). Arithmetic and algebra found their logical justification in geometry; hence algebra could not replace geometry or exist as its equal. The philosopher Thomas Hobbes (1588–1679), though only a minor figure in mathematics, nevertheless spoke for many mathematicians when he objected to the "whole herd of them who apply their algebra to geometry." Hobbes said that these algebraists mistook the symbols for geometry and characterized John Wallis's book on the conics as scurvy and as a "scab of symbols."

Despite the hindrances to appreciation of what Descartes and Fermat had contributed, a number of men gradually took up and expanded coordinate geometry. The first task was to explain Descartes's idea. A Latin translation of *La Géométrie* by Frans van Schooten (1615–60), first published in 1649 and republished several times, not only made the book available in

8. *Arithmetica Universalis*, 1707, p. 282.

the language all scholars could read but contained a commentary which expanded Descartes's compact presentation. In the edition of 1659–61, van Schooten actually gave the algebraic form of a transformation of coordinates from one base line (x-axis) to another. He was so impressed with the power of Descartes's method that he claimed the Greek geometers had used it to derive their results. Having the algebraic work, the Greeks, according to van Schooten, saw how to obtain the results synthetically—he showed how this could be done—and then published their synthetic methods, which are less perspicuous than the algebraic, to amaze the world. Van Schooten may have been misled by the word "analysis," which to the Greeks meant analyzing a problem, and the term "analytic geometry," which specifically described Descartes's use of algebra as a method.

John Wallis, in *De Sectionibus Conicis* (1655), first derived the equations of the conics by translating Apollonius' geometric conditions into algebraic form (much as we did in Chap. 4, sec. 12) in order to elucidate Apollonius' results. He then defined the conics as curves corresponding to second degree equations in x and y and proved that these curves were indeed the conic sections as known geometrically. He was probably the first to use equations to prove properties of the conics. His book helped immensely to spread the idea of coordinate geometry and to popularize treatment of the conics as curves in the plane instead of as sections of a cone, though the latter approach persisted. Moreover, Wallis emphasized the validity of the algebraic reasoning whereas Descartes, at least in his *Géométrie*, really rested on the geometry, regarding algebra as just a tool. Wallis was also the first to consciously introduce negative abscissas and ordinates. Newton, who did this later, may have gotten the idea from Wallis. We can contrast van Schooten's remark on method with one by Wallis, who said that Archimedes and nearly all the ancients so hid from posterity their method of discovery and analysis that the moderns found it easier to invent a new analysis than to seek out the old.

Newton's *The Method of Fluxions and Infinite Series*, written about 1671 but first published in an English translation by John Colson (d. 1760) under the above title in 1736, contains many uses of coordinate geometry, such as sketching curves from equations. One of the original ideas it offers is the use of new coordinate systems. The seventeenth- and even many of the eighteenth-century men generally used one axis, with the y-values drawn at an oblique or right angle to that axis. Among the new coordinate systems introduced by Newton is the location of points by reference to a fixed point and a fixed line through that point. The scheme is essentially our polar coordinate system. The book contains many variations on the polar coordinate idea. Newton also introduced bipolar coordinates. In this scheme a point is located by its distance from two fixed points (Fig. 15.7). Because this work of Newton did not become known until 1736, credit for the discovery of polar coordinates

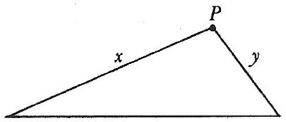

Figure 15.7

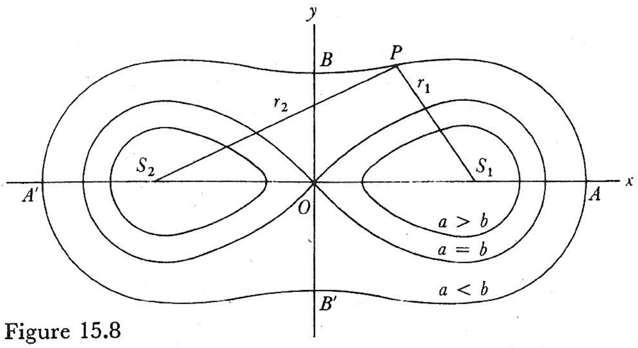

Figure 15.8

is usually given to James (Jakob) Bernoulli who published a paper on what was essentially this scheme in the *Acta Eruditorum* of 1691.

Many new curves and their equations were introduced. In 1694 Bernoulli introduced the lemniscate,[9] which played a major role in eighteenth-century analysis. This curve is a special case of a class of curves called the Cassinian ovals (general lemniscates) introduced by Jean-Dominique Cassini (1625–1712), though they did not appear in print until his son Jacques (1677–1756) published the *Eléments d'astronomie* in 1749. The Cassinian ovals (Fig. 15.8) are defined by the condition that the product $r_1 r_2$ of the distances of any point on the curve from two fixed points S_1 and S_2 equals b^2 where b is a constant. Let the distance $S_1 S_2$ be $2a$. Then if $b > a$ we get the non-self-intersecting oval. If $b = a$ we get the lemniscate introduced by James Bernoulli. And if $b < a$ we get the two separate ovals. The rectangular coordinate equation of the Cassinian ovals is of the fourth degree. Descartes himself introduced the logarithmic spiral,[10] which in polar coordinates has the equation $\rho = a^{\theta}$, and discovered many of its properties. Still other curves, among them the catenary and cycloid, will be noted in other connections.

The beginning of an extension of coordinate geometry to three dimensions was made in the seventeenth century. In Book II of his *Géométrie* Descartes remarks that his ideas can easily be made to apply to all those curves that can be conceived of as generated by the regular movements of a point in three-dimensional space. To represent such curves algebraically his plan is to drop perpendiculars from each point of the curve upon two planes

9. *Acta Erud.*, Sept. 1694 = *Opera*, 2, 608–12.
10. Letter to Mersenne of Sept. 12, 1638 = *Œuvres*, 2, 360.

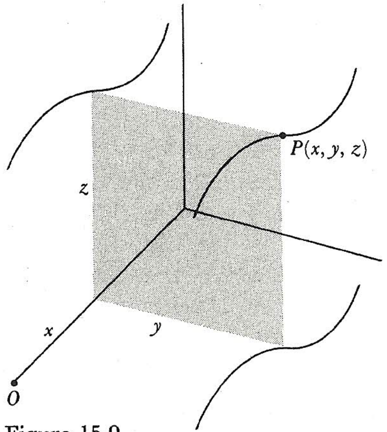

Figure 15.9

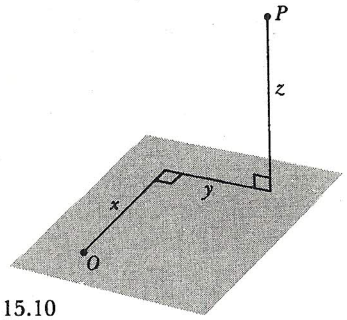

Figure 15.10

intersecting at right angles (Fig. 15.9). The ends of these perpendiculars will each describe a curve in the respective plane. These plane curves can then be treated by the method already given. Earlier in Book II Descartes observes that one equation in three unknowns for the determination of the typical point C of a locus represents a plane, a sphere, or a more complex surface. Clearly he appreciated thas his method could be extended to curves and surfaces in three-dimensional space, but he did not himself go further with the extension.

Fermat, in a letter of 1643, gave a brief sketch of his ideas on analytic geometry of three dimensions. He speaks of cyclindrical surfaces, elliptic paraboloids, hyperboloids of two sheets, and ellipsoids. He then says that, to crown the introduction of plane curves, one should study curves on surfaces. "This theory is susceptible of being treated by a general method which if I have leisure I will explain." In a work of half a page, *Novus Secundarum*,[11] he says that an equation in three unknowns gives a surface.

La Hire, in his *Nouveaux élémens des sections coniques* (1679), was a little more specific about three-dimensional coordinate geometry. To represent a surface, he first represented a point P in space by the three coordinates indicated in Figure 15.10 and actually wrote the equation of a surface. However, the development of three-dimensional coordinate geometry is the work of the eighteenth century and will be discussed later.

6. *The Importance of Coordinate Geometry*

In light of the fact that algebra had made considerable progress before Fermat and Descartes entered the mathematical scene, coordinate geometry

11. *Œuvres*, 1, 186–87; 3, 161–62.

was not a great technical achievement. For Fermat it was an algebraic rephrasing of Apollonius. With Descartes it arose as an almost accidental discovery as he continued the work of Vieta and others in expediting the solution of determinate construction problems by the introduction of algebra. But coordinate geometry changed the face of mathematics.

By arguing that a curve is any locus that has an algebraic equation, Descartes broadened in one swoop the domain of mathematics. When one considers the variety of curves that have come to be accepted and used in mathematics and compares this assemblage with what the Greeks had accepted, one sees how important it was that the Greek barriers be stormed.

Through coordinate geometry Descartes sought to introduce method in geometry. He achieved far more than he envisioned. It is commonplace today to recognize not only how readily one can prove, with the aid of the algebra, any number of facts about curves, but also that the method of approaching the problems is almost automatic. The methodology is even more powerful. When letters began to be used by Wallis and Newton to stand for positive and negative numbers and later even for complex numbers, it became possible to subsume under one algebraic treatment many cases that pure geometry would have had to treat separately. For example, in synthetic geometry, to prove that the altitudes of a triangle meet in a point, intersections inside and outside the triangle are considered separately. In coordinate geometry they are considered together.

Coordinate geometry made mathematics a double-edged tool. Geometric concepts could be formulated algebraically and geometric goals attained through the algebra. Conversely, by interpreting algebraic statements geometrically one could gain an intuitive grasp of their meanings as well as suggestions for the deduction of new conclusions. These virtues were cited by Lagrange in his *Leçons élémentaires sur les mathématiques*:[12] "As long as algebra and geometry travelled separate paths their advance was slow and their applications limited. But when these two sciences joined company, they drew from each other fresh vitality and thenceforward marched on at a rapid pace towards perfection." Indeed the enormous power mathematics developed from the seventeenth century on must be attributed, to a very large extent, to coordinate geometry.

The most significant virtue of coordinate geometry was that it provided science with just that mathematical facility it had always sorely needed and which, in the seventeenth century, was being openly demanded—quantitative tools. The study of the physical world does seem to call primarily for geometry. Objects are basically geometrical figures, and the paths of moving bodies are curves. Indeed Descartes himself thought that all of physics could be reduced to geometry. But, as we have pointed out, the uses of science in geodesy, navigation, calendar-reckoning, astronomical predictions, projectile motion,

12. *Œuvres*, 7, 183–287, p. 271 in part.

and even the design of lenses, which Descartes himself undertook, call for quantitative knowledge. Coordinate geometry made possible the expression of shapes and paths in algebraic form, from which quantitative knowledge could be derived.

Thus algebra, which Descartes had thought was just a tool, an extension of logic rather than part of mathematics proper, became more vital than geometry. In fact, coordinate geometry paved the way for a complete reversal of the roles of algebra and geometry. Whereas from Greek times until about 1600 geometry dominated mathematics and algebra was subordinate, after 1600 algebra became the basic mathematical subject; in this transposition of roles the calculus was to be the decisive factor. The ascendancy of algebra aggravated the difficulty to which we have already called attention, namely, that there was no logical foundation for arithmetic and algebra; but nothing was done about it until the late nineteenth century.

The fact that algebra was built up on an empirical basis has led to confusion in mathematical terminology. The subject created by Fermat and Descartes is usually referred to as analytic geometry. The word "analytic" is inappropriate; coordinate geometry or algebraic geometry (which now has another meaning) would be more suitable. The word "analysis" had been used since Plato's time to mean the process of analyzing by working backward from what is to be proved until one arrives at something known. In this sense it was opposed to "synthesis," which describes the deductive presentation. About 1590 Vieta rejected the word "algebra" as having no meaning in the European language and proposed the term "analysis" (Chap. 13, sec. 8); the suggestion was not adopted. However, for him and for Descartes, the word "analysis" was still somewhat appropriate to describe the application of algebra to geometry because the algebra served to analyze the geometric construction problem. One assumed the desired geometric length was known, found an equation that this length satisfied, manipulated the equation, and then saw how to construct the required length. Thus Jacques Ozanam (1640–1717) said in his *Dictionary* (1690) that moderns did their analysis by algebra. In the famous eighteenth-century *Encyclopédie*, d'Alembert used "algebra" and "analysis" as synonyms. Gradually, "analysis" came to mean the algebraic method, though the new coordinate geometry, up to about the end of the eighteenth century, was most often formally described as the application of algebra to geometry. By the end of the century the term "analytic geometry" became standard and was frequently used in titles of books.

However, as algebra became the dominant subject, mathematicians came to regard it as having a much greater function than the analysis of a problem in the Greek sense. In the eighteenth century the view that algebra as applied to geometry was more than a tool—that algebra itself was a basic method of introducing and studying curves and surfaces (the supposed view of Fermat as opposed to Descartes)—won out, as a result of the work of Euler, Lagrange, and Monge. Hence the term "analytic geometry" implied proof as well as the use of the algebraic method. Consequently we now speak of analytic geometry as opposed to synthetic geometry, and we no longer mean that one is a method of invention and the other of proof. Both are deductive.

In the meantime the calculus and extensions such as infinite series entered mathematics. Both Newton and Leibniz regarded the calculus as an extension of algebra; it was the algebra of the infinite, or the algebra that dealt with an infinite number of terms, as in the case of infinite series. As late as 1797, Lagrange, in *Théorie des fonctions analytiques*, said that the calculus and its developments were only a generalization of elementary algebra. Since algebra and analysis had been synonyms, the calculus was referred to as analysis. In a famous calculus text of 1748 Euler used the term "infinitesimal analysis" to describe the calculus. This term was used until the late nineteenth century, when the word "analysis" was adopted to describe the calculus and those branches of mathematics built on it. Thus we are left with a confusing situation in which the term "analysis" embraces all the developments based on limits, but "analytic geometry" involves no limit processes.

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